Indian Statistical Institute
M.Math. II Year
First Semester Back Paper Exam 2006-07
Advanced Probability

Time: 3 hrs | Date: -01-07 | Max. Marks:90

Instructor: B Rajeev

(1) (a) Suppose X_1, X_2 are independent random variables, $EX_i = 0$ and $Var(X_i) < \infty$. For $\alpha > 0$ show that ,

$$P(\max(X_1, X_1 + X_2) \geq \alpha) \leq \frac{1}{\alpha^2} \text{ Var}(X_1 + X_2)$$

(10)

(b) Let (X_n) be an independent sequence , with each X_i having exponential distribution with parameter $\alpha > 0$. Let $M_n := max(X_1, \cdots, X_n)$. Let $Y_n := M_n - \frac{logn}{\alpha}$. Show that for all x in $I\!R$

$$\lim_{n\to\infty} P(Y_n \leq x) = e^{-e^{-\alpha x}}$$

(8)

(c) Let (X_n) be a sequence of random variables such that $X_n \Rightarrow a$ for some real number a. Suppose $h : \mathbb{R} \to \mathbb{R}$ is measurable and continuous at a. Show that $h(X_n) \Rightarrow h(a)$. (5)

(2) (a) Let μ be a probability measure on $\mathbb{R}$ and ϕ its characteristic function. For $u > 0$, show that

$$\mu\{x : |ux| > 2\} < \frac{1}{u} \int_{-u}^{u} (1 - \phi(t))\, dt$$

(5)

(b) Let (μ_n) be a sequence of probability measures on $\mathbb{R}^d$ and let (ϕ_n) be the corresponding sequence of characteristic functions. Suppose that $\phi_n(t) \to g(t)$ for all $t \in \mathbb{R}$, where $g(t)$ is continuous at $t = 0$. Then there exists a probability measure μ on $\mathbb{R}$ such that $\mu_n \Rightarrow \mu$ and $g(t)$ is the characteristic function of μ. (Hint : show that the sequence (μ_n) is tight) (7)

(c) Suppose μ and ν are two probability measures on $\mathbb{R}^k$ and for all α, $\mu\{x : \ t \cdot x \leq \alpha\} = \nu\{x : \ t \cdot x \leq \alpha\}$ for all $t = (t_1, \cdots, t_k)$. Show that $\mu = \nu$. (8)

(3) Let T be a non empty set and let $\mathbb{R}^T$ be the 'product space' $\mathbb{R}^T := \{x| \ x : T \to \mathbb{R}\}$. Let $\mathcal{R}^T$ be the sigma field generated by the coordinate random variables $\{\xi_t : t \in T\}$ and let $\mathcal{R}_0^T$ be the class of finite dimensional cylinder sets i.e subsets A of the form

$$A = \{x \in \mathbb{R}^T : (\xi_{t_1}(x), \cdots, \xi_{t_k}(x)) \in H\}$$

a) Show that $\mathcal{R}_0^T$ is a field (10)

b) Show that $\mathcal{R}^T$ is the smallest sigma field containing $\mathcal{R}_0^T$. (5)

(4) Let f be a non negative , measurable and integrable function on $\mathbb{R}$. Let $F(x) := \int_{-\infty}^{x} f(t)dt$. Using the fact that a non decreasing function has a finite derivative outside a set of Lebesgue measure zero, show that $F'(x) = f(x)$, almost everywhere with respect to Lebesgue measure. (10)

(5) Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathcal{G}$ be a sub sigma field of $\mathcal{F}$.

(a) let $A_n \in \mathcal{F}$ be disjoint sets for $n = 1, \cdots$. Show that

$$P(\bigcup_n A_n|\mathcal{G}) = \sum_n P(A_n|\mathcal{G}) \ a.s$$

(7)

(b) If $X$and Y are random variables such that X is $\mathcal{G}$ measurable , Y and XY are integrable then , show that

$$E[XY|\mathcal{G}] = XE[Y|\mathcal{G}] \ a.s$$

(10)

(c) Let $(X_n)_{n\geq 1}$ be an independent sequence of mean zero random variables. Let $\mathcal{F}_n := \sigma\{X_1, \cdots X_n\}$ and let $S_n = X_1 + \cdots + X_n$. Show that $(S_n, \mathcal{F}_n)_{n\geq 1}$ is a martingale. (5).